A high performance dual revised simplex solver

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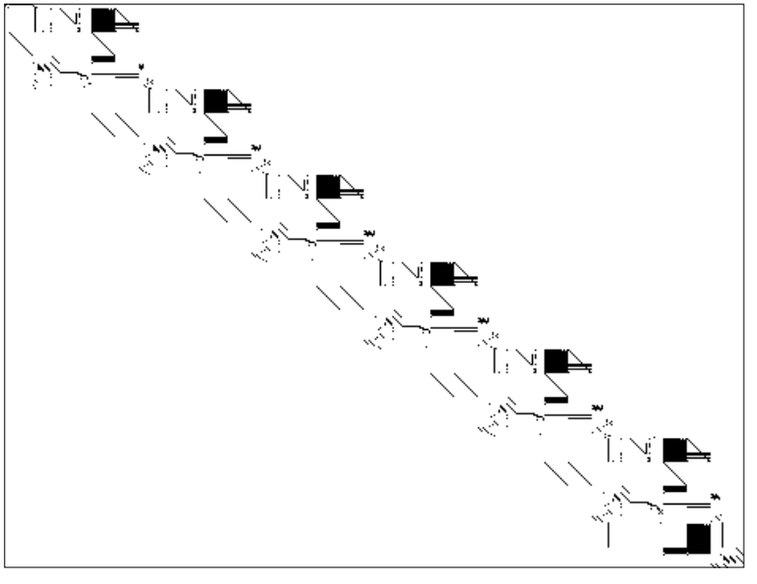
Overview

- LP problems and the dual simplex method
- Why use dual simplex method
- Why exploit parallelism
- Dual revised simplex with suboptimization
- Preliminary results
- Alternative product form update for GPU extension
- Conclusions

Linear programming (LP)

$$\begin{array}{ll} \text{minimize} & f = \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

- Fundamental model in optimal decision-making
- Solution techniques
  - Simplex method (1947)
  - Interior point methods (1984–date)
- Large problems have
  - 10^3–10^7 variables
  - 10^3–10^7 constraints
- Matrix A is (usually) sparse



STAIR: 356 rows, 467 columns and 3856 nonzeros

Mathematics of LP

$$\begin{array}{ll} \text{minimize} & f = \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad (P)$$

- Geometry:
  - Feasible points form a convex polyhedron
- Results:
  - An optimal solution occurs at a vertex
  - At a vertex the variable set can be partitioned as $\mathcal{B} \cup \mathcal{N}$ and constraints as

    $$B\boldsymbol{x}_B + N\boldsymbol{x}_N = \boldsymbol{b}$$

    so B is nonsingular and $\boldsymbol{x}_N = \boldsymbol{0}$
- Dual LP problem:

$$\begin{array}{ll} \text{maximize} & f = \boldsymbol{b}^T \boldsymbol{y} \\ \text{subject to} & A^T \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c} \quad \boldsymbol{s} \geq \boldsymbol{0} \end{array} \qquad (D)$$

- Result:
  - Optimal partition $\mathcal{B} \cup \mathcal{N}$ for (P) also solves (D)

The reduced LP problem

At a vertex, for a partition $\mathcal{B} \cup \mathcal{N}$ with B nonsingular and $\boldsymbol{x}_N = \mathbf{0}$, the original problem is

$$\begin{array}{lrcccc} \text{minimize} & f & = & \boldsymbol{c}_N^T \boldsymbol{x}_N & + & \boldsymbol{c}_B^T \boldsymbol{x}_B \\ \text{subject to} & & & N\,\boldsymbol{x}_N & + & B\,\boldsymbol{x}_B = \boldsymbol{b} \\ & & & \boldsymbol{x}_N \geq \mathbf{0} & & \boldsymbol{x}_B \geq \mathbf{0}. \end{array}$$

Eliminate $\boldsymbol{x}_B$ from the objective to give the reduced LP problem

$$\begin{array}{lrcccc} \text{minimize} & f & = & \boldsymbol{s}_N^T \boldsymbol{x}_N & + & \hat{f} \\ \text{subject to} & & & \hat{N}\,\boldsymbol{x}_N & + & I\,\boldsymbol{x}_B = \hat{\boldsymbol{b}} \\ & & & \boldsymbol{x}_N \geq \mathbf{0} & & \boldsymbol{x}_B \geq \mathbf{0}, \end{array}$$

where $\hat{\boldsymbol{b}} = B^{-1}\boldsymbol{b}$, $\hat{N} = B^{-1}N$, $\hat{f} = \boldsymbol{c}_B^T \hat{\boldsymbol{b}}$ and $\boldsymbol{s}_N$ is given by

$$\boldsymbol{s}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \hat{N}$$

Vertex is optimal $\iff$ $\boldsymbol{x}_B \geq \mathbf{0}$ and $\boldsymbol{s}_N \geq \mathbf{0}$

Primal vs dual simplex

Finding an optimal partition $\mathcal{B} \cup \mathcal{N}$ underpins the simplex method

- Primal simplex method
  - Maintains $\boldsymbol{x}_B \geq \mathbf{0}$
  - Moves along edges of the feasible region of (P)
  - Terminates when $\boldsymbol{s}_N \geq \mathbf{0}$
- Dual simplex method
  - Maintains $\boldsymbol{s}_N \geq \mathbf{0}$
  - Moves along edges of the feasible region of (D)
  - Terminates when $\boldsymbol{x}_B \geq \mathbf{0}$
- Adaptations of both are required to find initial feasible point

Implementation: dual standard simplex method

	$\mathcal{N}$	$\mathcal{B}$	RHS
1 $\vdots$ m	$\hat{N}$	I	$\hat{\boldsymbol{b}}$
0	$\boldsymbol{s}_N^T$	$\mathbf{0}^T$	$-\hat{f}$

In each iteration:

- Choose a row p with $\hat{b}_p < 0$
- Use the pivotal row p of $\hat{N}$ and $\boldsymbol{s}_N$ to find the pivotal column q with $\beta = s_q/\hat{a}_{pq}$
- Exchange indices p and q between $\mathcal{B}$ and $\mathcal{N}$
- Update tableau corresponding to this basis change

$$\begin{aligned} \hat{N} &:= \hat{N} - (1/\hat{a}_{pq})\hat{\boldsymbol{a}}_q\hat{\boldsymbol{a}}_p^T & \hat{\boldsymbol{b}} &:= \hat{\boldsymbol{b}} - (\hat{b}_p/\hat{a}_{pq})\hat{\boldsymbol{a}}_q \\ \boldsymbol{s}_N^T &:= \boldsymbol{s}_N^T - \beta\hat{\boldsymbol{a}}_p^T & -\hat{f} &:= -\hat{f} - \beta\hat{b}_p \end{aligned}$$

Implementation: dual revised simplex method

- Maintains a representation of B^{-1} (rather than explicit $\hat{N}$)
- Rows and columns of $\hat{N}$ obtained as required
- Pivotal row is

$$\hat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N, \text{ where } \boldsymbol{\pi}_p^T = \boldsymbol{e}_p^T B^{-1}$$

- Pivotal column is

$$\hat{\boldsymbol{a}}_q = B^{-1}\boldsymbol{a}_q, \text{ where } \boldsymbol{a}_q \text{ is column } q \text{ of } A$$

- Representation of B^{-1} is updated by exploiting

$$B := B + (\boldsymbol{a}_q - \boldsymbol{a}_p)\boldsymbol{e}_p^T$$

- Periodically a representation of B^{-1} is formed from scratch
- Efficient solution of large sparse LP problems requires the revised simplex method

Implementation: representing B^{-1}

- Factorize some basis matrix $B_0 = L_0 U_0$
  Store non-trivial columns of L_0 and U_0 as representation of B_0^{-1}
- Updating representation of B^{-1} each iteration exploits

$$\begin{aligned} B &:= B + (\boldsymbol{a}_q - \boldsymbol{a}_p)\boldsymbol{e}_p^T \\ &= B[I + B^{-1}(\boldsymbol{a}_q - \boldsymbol{a}_p)\boldsymbol{e}_p^T] \\ &= B[I + (\hat{\boldsymbol{a}}_q - \boldsymbol{e}_p)\boldsymbol{e}_p^T] \end{aligned}$$

  where $\hat{\boldsymbol{a}}_q$ is the pivotal column so, using Sherman-Morrison and $\hat{a}_{pq} = \boldsymbol{e}_p^T \hat{\boldsymbol{a}}_q$,

$$B_k^{-1} = \left[I - \frac{(\hat{\boldsymbol{a}}_q - \boldsymbol{e}_p)\boldsymbol{e}_p^T}{\hat{a}_{pq}} \right] B_{k-1}^{-1} = E_k^{-1} B_{k-1}^{-1}$$

- Hence

$$B_k^{-1} = H_k^{-1} B_0^{-1} \quad \text{where} \quad H_k^{-1} = E_k^{-1} \dots E_1^{-1}$$

Summary: major computational components for simplex implementations

Standard simplex method (SSM)

- Update tableau $\hat{N} := \hat{N} - (1/\hat{a}_{pq})\hat{\boldsymbol{a}}_q\hat{\boldsymbol{a}}_p^T$

Revised simplex method (RSM)

- Operations
  - Form $\boldsymbol{\pi}_p^T = \boldsymbol{e}_p^T B^{-1}$
  - Form $\hat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$
  - Form $\hat{\boldsymbol{a}}_q = B^{-1}\boldsymbol{a}_q$
- Inversion of B

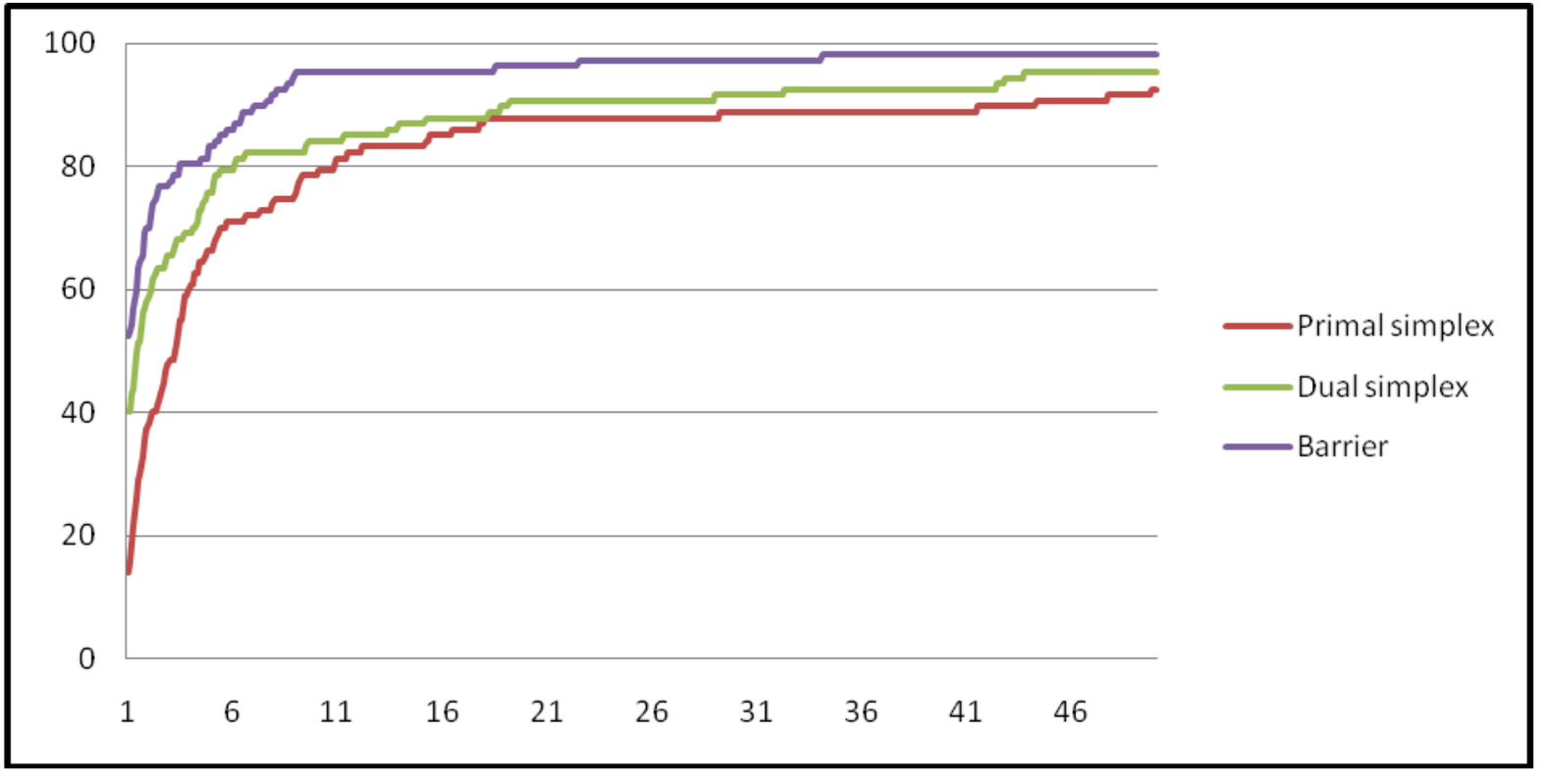
- Distinctive features
  - Vectors $\boldsymbol{e}_p$, $\boldsymbol{a}_q$ are always sparse
  - B may be highly reducible
  - B^{-1} may be sparse
  - Vectors $\boldsymbol{\pi}_p$, $\hat{\boldsymbol{a}}_p$ and $\hat{\boldsymbol{a}}_q$ may be sparse
- Efficient implementations must exploit these features

  H and McKinnon (1998–2005)

Why use the dual simplex method?

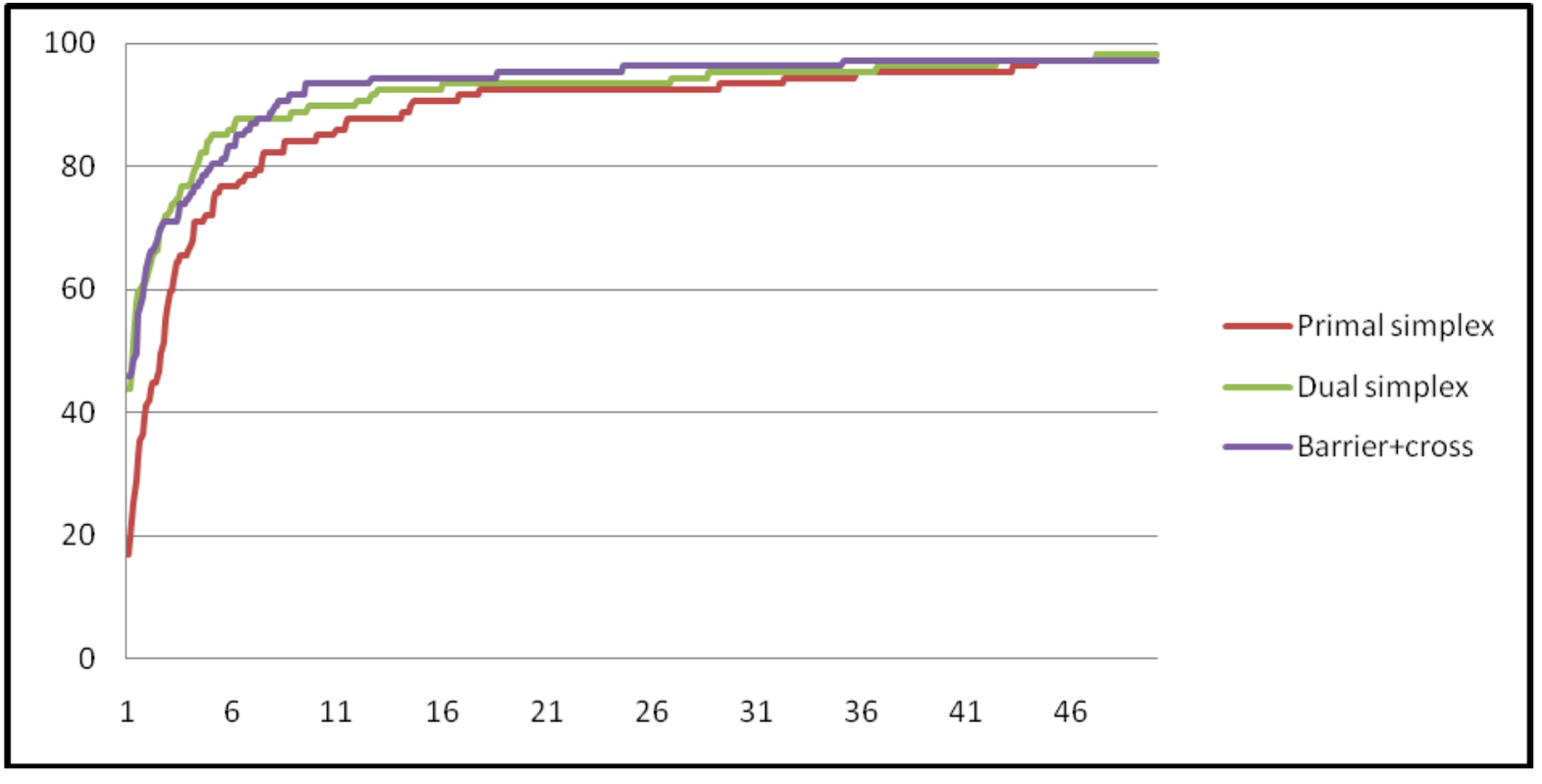
- Dual simplex method has been around almost as long as the primal simplex method
- More of theoretical interest until 1990's
- Now preferred to primal simplex
  - Easier to find feasible point of (D) to start
  - Has some efficient algorithmic tricks not available to primal
  - Dual feasibility retained when constraints are added (MIP)
- Primal simplex method applied to (D) can be made computationally equivalent

CPLEX LP solvers applied to standard test problems



- Dual simplex better than primal but barrier clearly better

CPLEX LP solvers applied to standard test problems



- Little to choose between dual simplex and barrier with crossover

Parallel simplex: why?

- Moore's law drives core counts per processor, but clock speeds will stabilise
- Serial performance of simplex is spectacularly good
  - Flop count per iteration is near optimal
  - Number of iterations is near optimal
- Can't wait for faster serial processors or algorithmic improvement
- Simplex method must try to exploit parallelism

Parallel simplex: immediate scope

- Standard simplex method
  - Update tableau $\hat{N} := \hat{N} - (1/\hat{a}_{pq})\hat{\boldsymbol{a}}_q\hat{\boldsymbol{a}}_p^T$

  Level 2 BLAS with $\hat{N}$ dense so "massively parallel"
- Revised simplex method
  - Operations $\boldsymbol{\pi}_p^T = \boldsymbol{e}_p^T B^{-1}$ and $\hat{\boldsymbol{a}}_q = B^{-1}\boldsymbol{a}_q$ are "inherently serial"
  - Operation $\hat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$ is "massively parallel"

  Amdahl's law implies little immediate scope for exploiting parallelism

Parallel simplex: nature of B_0^{-1} and H^{-1}

Properties of B_0^{-1} and H^{-1} are a majr influence on parallel schemes

- B_0^{-1} requires many vectors but they are short $O(1)$
  - Parallelism is fine grained
  - Unsuitable for data parallelism
  - Aim for task parallelism
- H^{-1} requires few vectors but they are long $O(m)$
  - Operations with H^{-1} can be posed as a sparse matrix-vector product
  - Suitable for data parallelism

Parallel simplex: past work

- **Data parallel standard simplex method**
  - Good parallel efficiency *was* achieved
  - Totally uncompetitive with serial revised simplex method without prohibitive resources
- **Data parallel revised simplex method**
  - Only immediate parallelism is in forming $\boldsymbol{\pi}_p^T N$
  - When $n \gg m$, cost of $\boldsymbol{\pi}_p^T N$ dominates: significant speed-up *was* achieved Bixby and Martin (2000)
- **Task parallel revised simplex method**
  - Overlap computational components for different iterations Wunderling (1996), H and McKinnon (1995-2005)
  - Modest speed-up *was* achieved on general sparse LP problems
- All data and task parallel implementations compromised by serial inversion of B

Review: H (2010)

Architectures: CPU or GPU or both?

Heterogeneous desk-top architectures

CPU:

- Fewer, faster cores
- Relatively slow memory transfer
- Welcomes algorithmically complex code
- Full range of development tools

GPU:

- More, slower cores
- Relatively fast memory transfer
- Global communication is expensive/difficult
- Very limited development tools

CPU and GPU:

- Possibly combine CPU and GPU to harness full computing power

Scope for task and data parallelism via suboptimization

- Perform multiple pricing—standard simplex suboptimization
  - Primal: Orchard-Hays (1968)
  - Dual: Rosander (1975)
- Algorithmically
  - Primal: Identify attractive column slice of tableau
  - Dual: Identify attractive row slice of tableau
  - Both perform standard simplex iterations to identify a set of basis changes
- Computationally
  - Solve systems with multiple RHS
  - Update tableaux
  - Form matrix products with multiple vectors
- Attractive in the days when memory access was expensive...

Primal: Parallel implementations by Wunderling (1996), H and McKinnon (1995-2005)

Dual: New, even in serial?

Dual revised simplex method with suboptimization

Let the current basis be B_k

- Choose attractive row set $\mathcal{P}$
- Form rows $\mathcal{P}$ of B_k^{-1}: $\hat{\boldsymbol{\pi}}_{\mathcal{P}}^T = \boldsymbol{e}_{\mathcal{P}}^T B_k^{-1}$
- Form rows $\mathcal{P}$ of tableau: $\hat{\boldsymbol{a}}_{\mathcal{P}}^T = \boldsymbol{\pi}_{\mathcal{P}}^T N_k$
- Perform l dual standard simplex iterations on $\hat{\boldsymbol{a}}_{\mathcal{P}}^T$ to identify column set $\mathcal{Q}$
- Form pivotal columns for $q \in \mathcal{Q}$: $\hat{\boldsymbol{a}}_{\mathcal{Q}} = B_k^{-1} \boldsymbol{a}_{\mathcal{Q}}$
- Use $\hat{\boldsymbol{a}}_{\mathcal{Q}}$ to update $\hat{\boldsymbol{b}}_k$ and H_k^{-1} to obtain $\hat{\boldsymbol{b}}_{k+l}$ and representation of B_{k+l}^{-1}

Parallel operations with B_k^{-1}

- Consider operations with $B_k^{-1} = H_k^{-1} B_0^{-1}$ in stages corresponding to B_0 and H_k
- Form $\hat{\boldsymbol{\pi}}_{\mathcal{P}}^T = \boldsymbol{e}_{\mathcal{P}}^T B_k^{-1}$ as

$$\tilde{\boldsymbol{\pi}}_{\mathcal{P}}^T = \boldsymbol{e}_{\mathcal{P}}^T H_k^{-1}$$

  then

$$\hat{\boldsymbol{\pi}}_{\mathcal{P}}^T = \tilde{\boldsymbol{\pi}}_{\mathcal{P}}^T B_0^{-1}$$

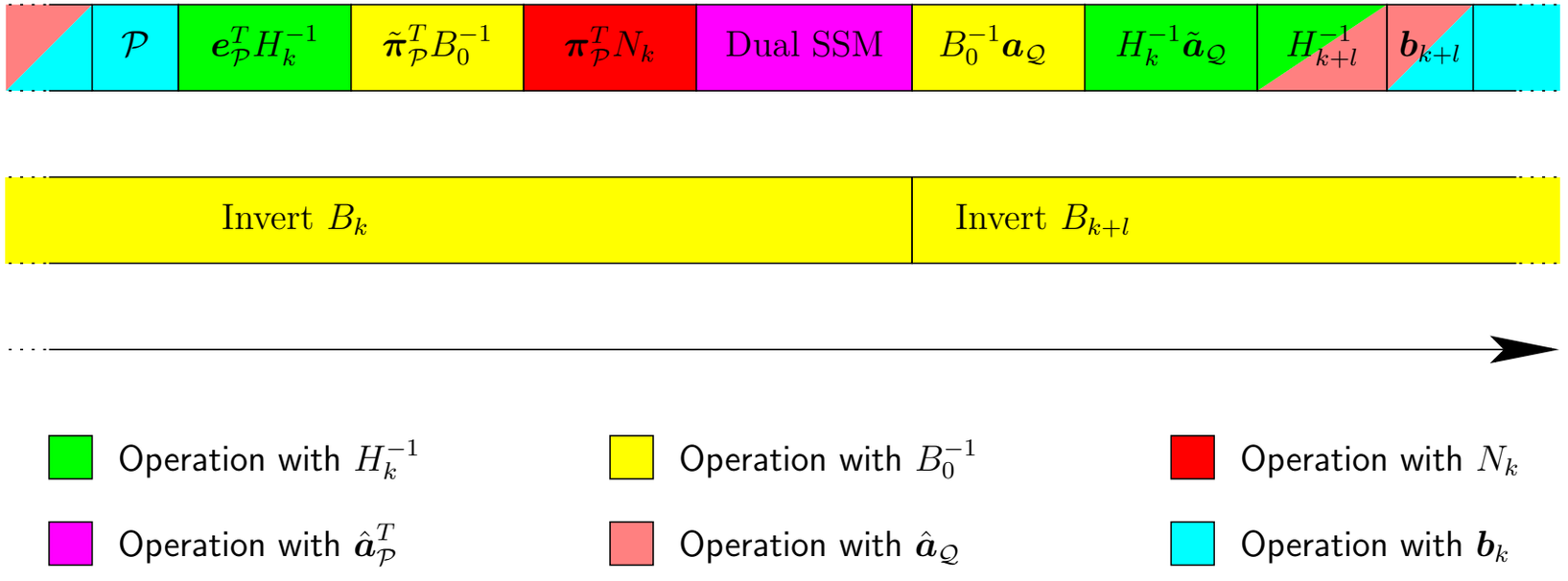
- Form $\hat{\boldsymbol{a}}_{\mathcal{Q}} = B_k^{-1} \boldsymbol{a}_{\mathcal{Q}}$ as

$$\tilde{\boldsymbol{a}}_{\mathcal{Q}} = B_0^{-1} \boldsymbol{a}_{\mathcal{Q}}$$

  then

$$\hat{\boldsymbol{a}}_{\mathcal{Q}} = H_k^{-1} \tilde{\boldsymbol{a}}_{\mathcal{Q}}$$

Gantt chart of computation and data



- Inversion of B_k can take a whole major iteration...
  - ... but not longer—for reasons of numerical stability
  - Serial inversion could lead to load imbalance
  - Inversion could be parallel?

Prototype implementation

- Written by Qi Huangfu (2010)
- Uses (generally) highly efficient core routines
  - Hyper-sparse matrix-vector product $\boldsymbol{\pi}_{\mathcal{P}}^{T} N_k$
  - Dual steepest edge pricing
- Does not (yet) use
  - Hyper-sparse operations with B^{-1}
  - Bound-flipping ratio test
- Written in C++ with OpenMP using the Intel C++ compiler
- Tested on a dual quad-core AMD Opteron 2378 system
- Uses one pivotal row per core used
- First run last week!

Preliminary results

pds-06: 9882 rows, 28655 columns and 82269 nonzeros

	Cores				
	1	2	4	8	clp dual
Major iterations	10266	5049	2543	1253	9808
Total iterations	10266	9625	8820	7616	9808
Solution time (s)	3.76	2.51	2.00	1.52	1.92
Speed (iter/s)	2730	3836	4419	5017	5111

Preliminary results

pds-10: 16559 rows, 48763 columns and 140063 nonzeros

	Cores				
	1	2	4	8	clp dual
Major iterations	17983	9158	4807	2557	17713
Total iterations	17983	17051	16263	15404	17713
Solution time (s)	12.58	10.01	6.86	5.72	6.61
Speed (iter/s)	1430	1704	2370	2695	2682

Can the simplex method exploit a GPU?

- Standard simplex method implemented in single precision by Smith as **i8**
- Experiments on machine with two quad-core AMD CPUs and GTX285 NVIDIA GPU
- For **dense** LP problems: best results

Solver	Type	HPC	Time	Iterations	Speed (iter/s)
gurobi	primal RSM	serial	1357	16034	12
gurobi	dual RSM	serial	976	14518	15
i6	primal SSM	parallel	4039	288419	79
i8	primal SSM	GPU	800	221157	276

- Not (really) of practical value but a good learning exercise
- No hope of beating serial solvers on sparse LP problems
- Now adding steepest edge and extending to double precision for Tesla C2070

GPU extension

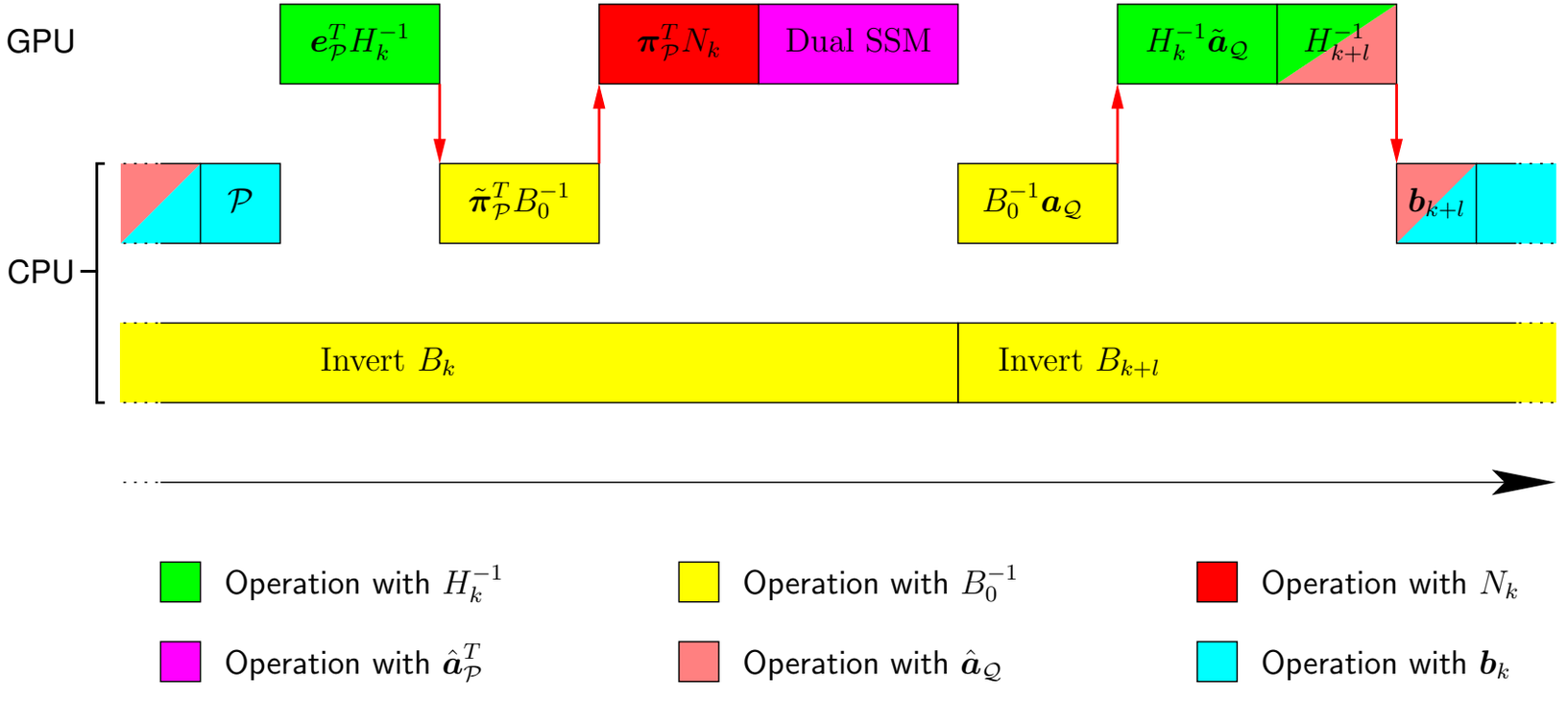
- Put dual standard simplex iterations on a GPU
- Consider putting other simple "rectangular" operations on a GPU
  - Sparse matrix-vector product $\boldsymbol{\pi}_{\mathcal{P}}^T N_k$

    GPU out-performs comparable CPU for full vector (eg) Vuduc *et al.* (2010)

    Nvidia interested in case of sparse vector
  - Operations with H_k^{-1}

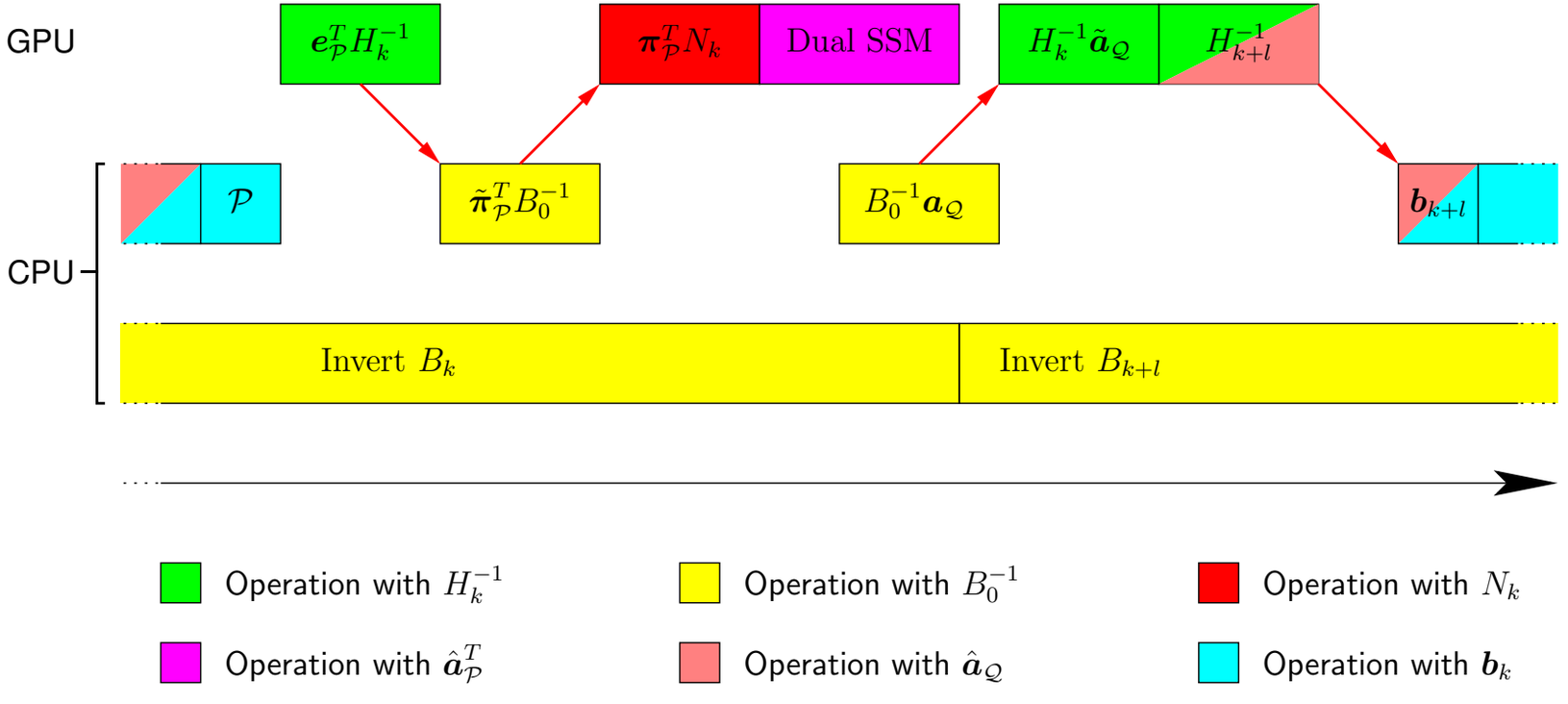
    Can be posed as a sparse matrix-vector product
- Need to limit data transfer between main memory and GPU

Splitting work between CPU and GPU



- CPU and GPU operations can be overlapped
- Data traffic between main memory and GPU is high

Overlapping CPU/GPU computation and communication



- Lucky if overlaping computation accommodates communication

Alternative product form update

- Little-known (unknown?) alternative product form update may offer a solution
- Updating representation of B^{-1} each iteration exploits

$$\begin{aligned} B &:= B + (\boldsymbol{a}_q - \boldsymbol{a}_p)\boldsymbol{e}_p^T \\ &= [I + (\boldsymbol{a}_q - \boldsymbol{a}_p)\boldsymbol{e}_p^T B^{-1}]B \\ &= [I + (\boldsymbol{a}_q - \boldsymbol{a}_p)\boldsymbol{\pi}_p^T]B \end{aligned}$$

so, using Sherman-Morrison,

$$B_k^{-1} = B_{k-1}^{-1}\left[I - \frac{(\boldsymbol{a}_q - \boldsymbol{a}_p)\boldsymbol{\pi}_p^T}{\hat{a}_{pq}}\right] = B_{k-1}^{-1}E_k^{-1}$$

- Hence reversed the order of inverse and update in representation of B_k^{-1}

$$B_k^{-1} = B_0^{-1}H_k^{-1} \quad \text{where} \quad H_k^{-1} = E_1^{-1}\ldots E_k^{-1}$$

Alternative dual revised simplex method with suboptimization

- Parallel operations with $B_k^{-1} = B_0^{-1} H_k^{-1}$
- Form $\hat{\boldsymbol{\pi}}_{\mathcal{P}}^T = \boldsymbol{e}_{\mathcal{P}}^T B_k^{-1}$ as

$$\tilde{\boldsymbol{\pi}}_{\mathcal{P}}^T = \boldsymbol{e}_{\mathcal{P}}^T B_0^{-1}$$

  then

$$\hat{\boldsymbol{\pi}}_{\mathcal{P}}^T = \tilde{\boldsymbol{\pi}}_{\mathcal{P}}^T H_k^{-1}$$

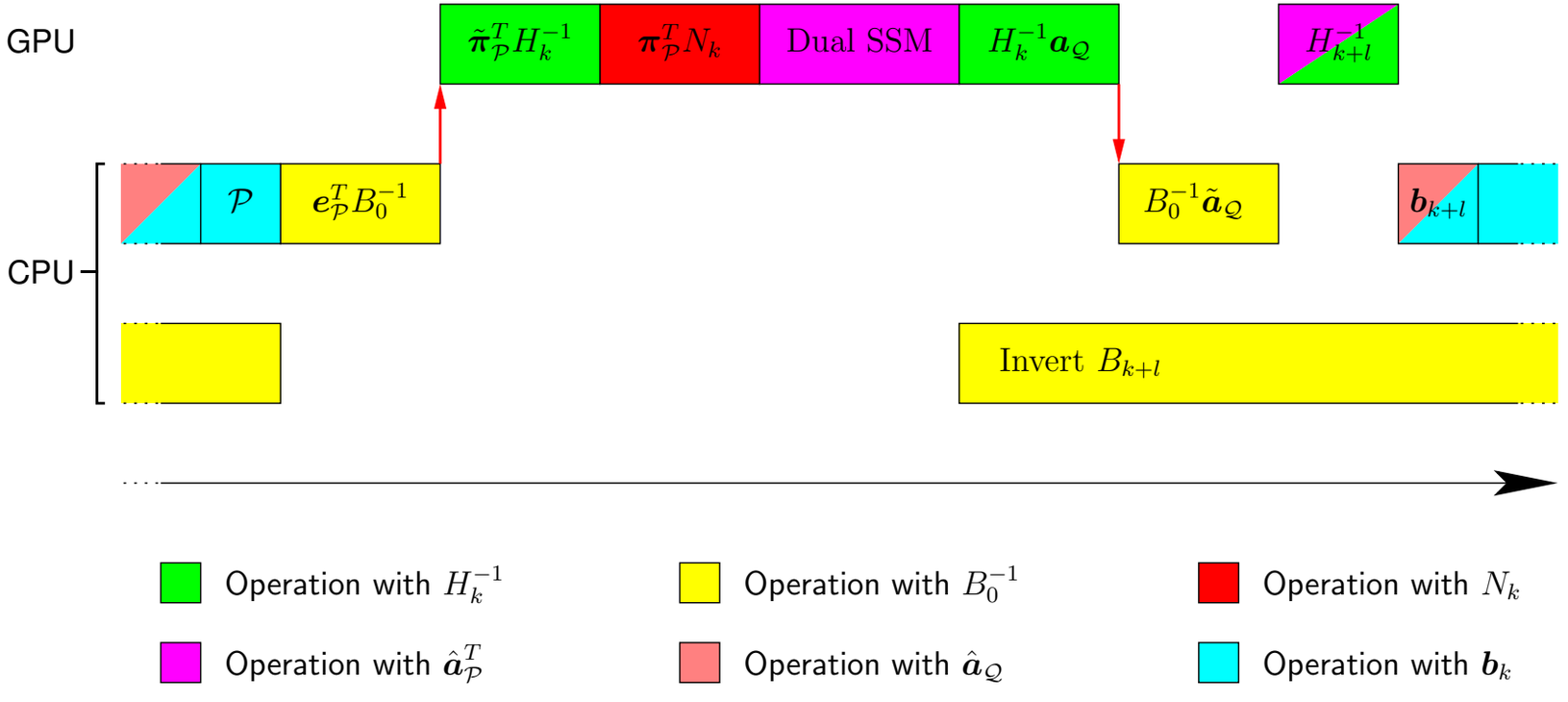
- Form $\hat{\boldsymbol{a}}_{\mathcal{Q}} = B_k^{-1} \boldsymbol{a}_{\mathcal{Q}}$ as

$$\tilde{\boldsymbol{a}}_{\mathcal{Q}} = H_k^{-1} \boldsymbol{a}_{\mathcal{Q}}$$

  then

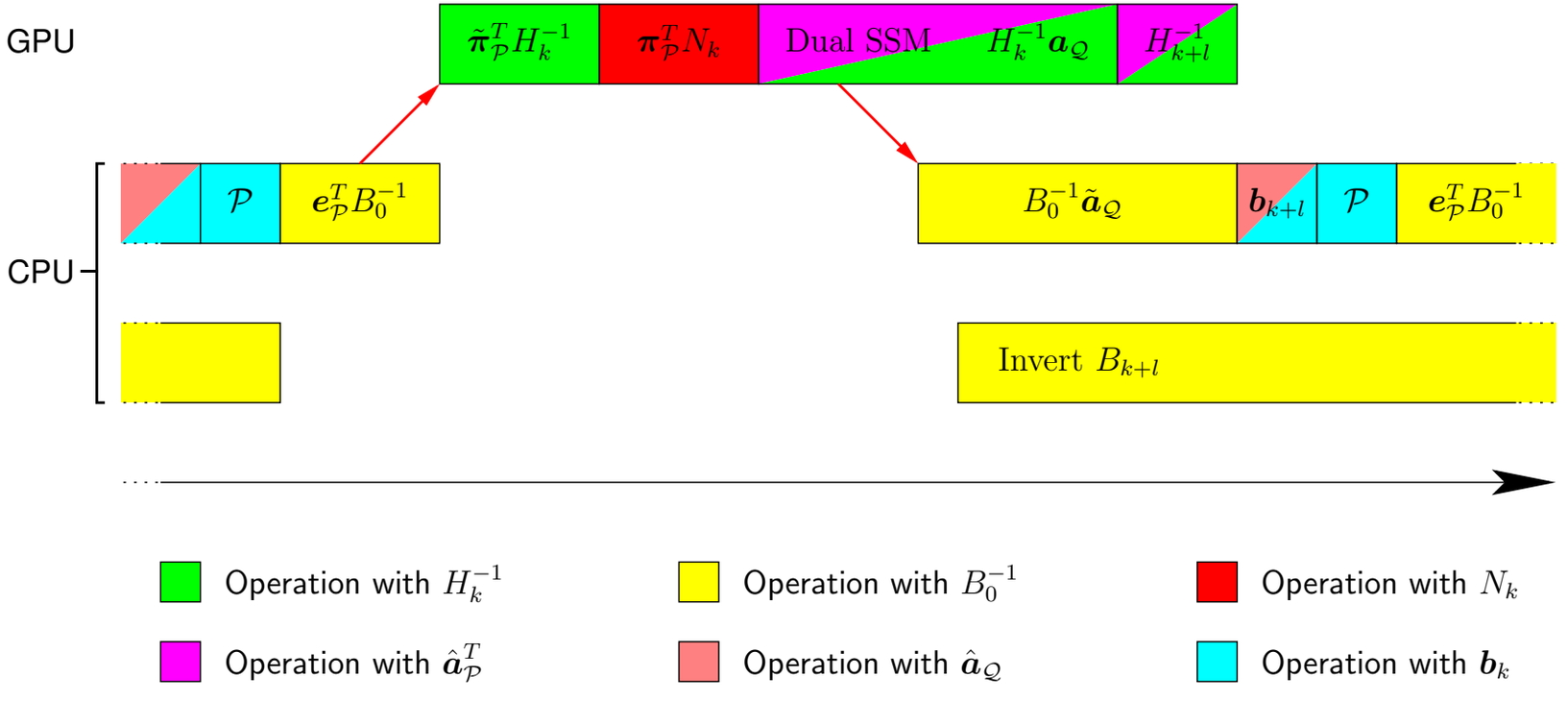
$$\hat{\boldsymbol{a}}_{\mathcal{Q}} = B_0^{-1} \tilde{\boldsymbol{a}}_{\mathcal{Q}}$$

Splitting work between CPU and GPU



- Data traffic between main memory and GPU is lower
- Shorter time for inversion of B_{k+l}

Overlapping CPU/GPU computation and communication



- Greater scope for overlapping computation
- Implementation will be **very** difficult

Conclusions

- Identified need for simplex method to exploit parallelism
- Developed prototype high performance dual revised simplex solver
- Initial results are encouraging
- Shown that the standard simplex method will run fast on a GPU
- Standard approach to linear algebra bad for CPU-GPU combination
- Alternative product form update may offer a solution

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